

Lecture 16

Today:

- * Heuristics for Brenier's theorem
- * The c -exponential map
- * Rademacher's theorem

Heuristics for Brenier's Theorem

(or: how to characterize optimal (ϕ, ψ) ??)

Consider a pair (ϕ, ψ) solving the Dual Problem, with $\phi = \psi^c$, $\psi = \phi^c$.

Fix $\eta \in C^0(Y)$, and set

$$\psi_s(y) = \psi(y) + s\eta(y)$$

Then $((\psi_s)^c, \psi_s)$ is an admissible pair

for every s ... What is the value of

$$\left. \frac{d}{ds} \left(\int_X (\psi_s)^c(x) d\mu(x) + \int_Y \psi_s(y) d\nu(y) \right) \right|_{s=0}$$

(Note: the point is whatever this is, it equals zero!)

Clearly,

$$\frac{d}{ds} \int_{\gamma} \psi_s(\gamma) d\gamma(\gamma) = \int_{\gamma} \eta(\gamma) d\gamma(\gamma)$$

The ψ_s^c is more delicate. What are the possible values of $\frac{d}{ds} \left| (\psi_s^c)^c(x) \right|_{s=0}$?

Working assumption: $(\psi^c)^c(x)$ is such that for every x there exists a unique $\gamma(x)$ such that

$$\argmin_{\gamma} \{ c(x, \gamma) - \psi(\gamma) \} = \{ \gamma(x) \}$$

In fact, let's assume^{*} an even stronger form of this assumption: $\exists \epsilon \in (-\epsilon, \epsilon)$ ($\epsilon > 0$) and every x there is a unique $\gamma_s(x)$ such that

(* this not needed, as we shall see)

$$\argmin_{\gamma} \{ c(x, \gamma) - \psi_s(\gamma) \} = \{ \gamma_s(x) \}$$

In particular, $\forall s \in (-\varepsilon, \varepsilon)$, $\forall x \in X$ we have

$$(\psi_s)^c(x) = c(x, y_s(x)) - \psi(y_s(x)) - s\eta(y_s(x))$$

Key fact: As $s \rightarrow 0$, for any fixed x ,

$$\left. \frac{d}{ds} \right|_{s=0} (c(x, y_s(x)) - \psi(y_s(x))) \geq 0$$

Indeed, for a given x , the map

$$y \mapsto c(x, y) - \psi(y)$$

is achieving its minimum at $y(x)$. Thus

$$c(x, y_s(x)) - \psi(y_s(x)) \geq c(x, y) - \psi(y)$$

ie. $s \mapsto c(x, y_s(x)) - \psi(y_s(x))$ achieves its minimum at 0. It follows that

$$\lim_{s \rightarrow 0} \frac{1}{s} \left((\psi_s)^c(x) - \psi(x) \right) \geq -\eta(y(x))$$

This used implicitly that as $s \rightarrow 0$ we have $y_s(x) \rightarrow y(x)$ (this follows from the working assumption)

Exercise: Prove this last claim.

(Hint: Show that

$$\lim_{s \rightarrow 0} (\psi_s)^c(x) = c(x, y(x)) - \psi(y(x))$$

and use the fact that there is a unique minimizer of $y \mapsto c(x, y) - \psi(y)$ to conclude that $y_s(x) \rightarrow y(x)$ as $s \rightarrow 0$.)

If all of the above holds, then we expect that

$$\begin{aligned} 0 &= \frac{d}{ds} \Big|_{s=0} \left(\int_X (\psi_s)^*(x) d\mu + \int_Y \psi_s(y) d\nu \right) \\ &\geq - \int_X \eta(y(x)) d\mu + \int_Y \eta(y) d\nu \end{aligned}$$

or,

$$\int_X \eta(y(x)) d\mu \geq \int_Y \eta(y) d\nu$$

Since $\eta(y) \in C^0(Y)$ was arbitrary, it follows that

$$\int_X \eta(y(x)) d\mu(x) = \int_Y \eta(y) d\nu(y).$$

The significance of what we just discussed is that if we knew that for each x we had

$$(\partial^c \phi)(x) = \{y(x)\}$$

then $y(x)$ has to be a Brenier solution of the Monge-problem.

This raises some questions:

- ① When is $y(x)$ even well-defined?
- ② How regular is it?

Well, these questions can be answered at least μ -a.e. provided there is some differentiability (i.e. X, Y are manifolds, $c(x, y)$ is smooth) and some regularity of the target measure. Then, Rademacher's theorem shall come to

the rescue!